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DYNAMIC ANALYSIS OF INTERACTING COPLANAR CRACKS IN A HALF SPACE WITH A CLAMPED BOUNDARY CONDITION USING BOUNDARY INTEGRAL EQUATIONS

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The three-dimensional dynamic problem of coplanar circular cracks in an elastic half-space with a clamped boundary condition is considered. The crack faces are subjected to harmonic loads. The problem is reduced to a system of two-dimensional boundary integral equations of the type of the Helmholtz potential for unknown discontinuities in the displacements of the opposite faces of the cracks. The stress intensity factors at the crack contours are obtained and discussed.

Key words: *elastic half-space, clamped boundary, plane cracks, steady vibrations, boundary integral equations.*

Introduction. It is well known that the strength of real bodies depends heavily on the presence of structural defects such as cracks and inclusions, which are often stress concentrators. In studies of the stress–strain state of bodies with defects, particular attention is paid to the inertia effects caused by dynamic loads. Thus, in the case of infinite and semi-infinite bodies with cracks under harmonic and impact loading, the stress concentrations arising near the defects can far exceed the static values [1–4]. The above-mentioned effects are also influenced by the presence of the outer surface of the body [5–7]. One of the effective methods for solving dynamic problems of three-dimensional theory of elasticity is the method of boundary integral equations (BIE) [8–13]. In the present paper, the BIE method is used to study the inertia effects near the contours of circular cracks in a half-space subjected to harmonic loads with a clamped boundary condition.

Formulation of the Problem. We consider an isotropic elastic half-space whose boundary surface S_0 is clamped. The half-space contains K plane circular cracks of radius a, which occupy regions S_k $(k = \overline{1, K})$ and are at equal depths $d = |O_0O_1|$ in a plane perpendicular to the boundary S_0. The opposite faces of the cracks $S_k^{\pm}$ are loaded by self-equilibrated harmonic tearing forces

$$N_{3k}^{+}(x_k, t) = -N_{3k}^{-}(x_k, t) = N_{3k}(x_k) \exp(-i\omega t) \qquad (k = \overline{1, K}),$$

where ω is the frequency of the applied force, t is time, $N_{3k}(x_k)$ is the amplitude of the force, and $i = \sqrt{-1}$. The opposite crack faces are not in contact. This condition can be satisfied if additional tensile forces are applied at infinity. We choose local Cartesian coordinate systems $O_kx_{1k}x_{2k}x_{3k}$ $(k = \overline{0, K})$ in such a manner that the domain $x_{30} \leqslant 0$ corresponds to the half-space and the planes $O_kx_{1k}x_{2k}$ contain the crack domains S_k (Fig. 1).

Since the loads vary harmonically in time, all characteristics of the wave field in the body vary with frequency ω. The problem of determining the stress–strain state of the half-space with defects reduces to the differential equation

$$\omega_1^{-2}\nabla(\nabla \cdot \boldsymbol{u}) - \omega_2^{-2}\nabla \times (\nabla \times \boldsymbol{u}) + \boldsymbol{u} = 0 \qquad (1)$$

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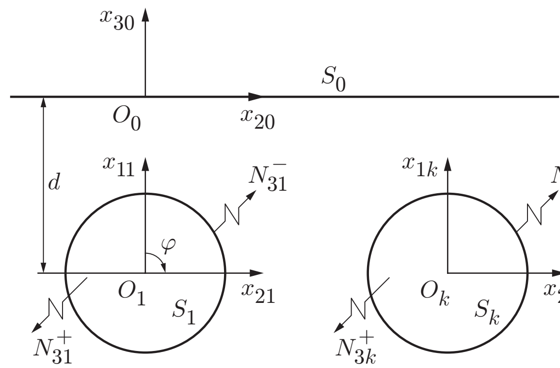

Fig. 1. Geometry of the problem.

subject to the boundary conditions

$$
\begin{gathered}
u_j(x_0) = 0, \qquad x_0 \in S_0, \qquad j = \overline{1,3}, \\
\sum_{j=1}^{3} \sigma_{j3k}(x_k)\cos(x_{3k}, x_{jk}) = N_{3k}(x_k), \qquad x_k \in S_k, \qquad k = \overline{1,K}, \\
N_{1k}(x_k) = N_{2k}(x_k) = 0, \qquad k = \overline{1,K}.
\end{gathered} \tag{2}
$$

Here $\boldsymbol{u}(u_1, u_2, u_3)$ and σ_{j3k} are the amplitudes of the displacement vector and stress-tensor components, respectively, $\nabla = \nabla(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\omega_j = \omega/c_j$ $(j = 1, 2)$, and c_1 and c_2 are the propagation velocities of the longitudinal and transverse waves, respectively, such that $c_2^2 = \gamma^2 c_1^2$, where $\gamma^2 = (1 - 2\mu)/(2(1-\mu))$ (μ is Poisson's ratio).

Construction of the Solution. We denote the coordinates of the point x_k in the kth coordinate system by x_{1k}, x_{2k}, and x_{3k} $(k = \overline{0,K})$ and the coordinate of the same point x_{kn} in the nth coordinate system by x_{1kn}, x_{2kn}, and x_{3kn} $(n = \overline{0,K})$. The following relations hold:

$$
x_{1kn} = x_{1k}, \qquad x_{2kn} = d_{kn}\cos(d_{kn}, x_{2k}) + x_{2k}, \qquad d_{kn} = |O_k O_n|, \qquad n,k = \overline{1,K}.
$$

According to the superposition principle, the displacements at an arbitrary point of the body are equal to the sum of the displacements u_{j0} $(j = \overline{1,3})$ from the half-space boundary S_0 and the displacements u_{jk} produced by opening of the opposite faces $S_k^{\pm}$ of the cracks:

$$
u_j(x_0) = u_{j0}(x_0) + \sum_{k=1}^{K} [\delta_{j1} u_{2k}(x_{k0}) + \delta_{j2} u_{3k}(x_{k0}) + \delta_{j3} u_{1k}(x_{k0})].
$$

Here δ_{ji} is the Kronecker symbol and the displacements $u_{jk}(x_k)$ are written in the form of integral representations [8]

$$
\begin{gathered}
u_{jk}(x_k) = -\frac{\partial P_{3k}^{(1)}}{\partial x_{jk}} + (1 - \delta_{j3})\Big[2\,\frac{\partial P_{3k}^{(2)}}{\partial x_{jk}} + \frac{\partial P_{jk}^{(2)}}{\partial x_{3k}}\Big] \\
+ \frac{2}{\omega_2^2}\,\frac{\partial}{\partial x_{jk}} \sum_{m=1}^{3} \Big[\delta_{m3}\Delta_k + (1 - \delta_{m3})\,\frac{\partial^2}{\partial x_{3k}\,\partial x_{mk}}\Big] \sum_{l=1}^{2} (-1)^{l+1} P_{mk}^{(l)}, \qquad k = \overline{0,K},
\end{gathered} \tag{3}
$$

where $\Delta_k = \dfrac{\partial^2}{\partial x_{1k}^2} + \dfrac{\partial^2}{\partial x_{2k}^2}$ is a two-dimensional Laplace operator, $P_{jk}^{(l)}(x_k) = \displaystyle\iint_{S_k} \Delta u_{jk}(\xi)\Phi_l(x_k,\xi)\,dS_\xi$ $(j = \overline{1,3}$, $l = 1, 2)$ are Helmholtz potentials, $\Phi_l(x_k,\xi) = \dfrac{\exp(i\omega_l|x_k - \xi|)}{|x_k - \xi|}$, and $|x_k - \xi| = \Big[\displaystyle\sum_{n=1}^{2}(x_{nk} - \xi_n)^2\Big]^{1/2}$.

The unknown densities Δu_{jk} $(j = \overline{1,3},\ k = \overline{1,K})$ of the potentials $P_{jk}^{(l)}(x_k)$ characterize the discontinuities in the displacements of the opposite faces of the cracks in the direction $O_k x_{jk}$:

$$
\Delta u_{jk}(x_k) = (u_{jk}^{+}(x_k) - u_{jk}^{-}(x_k))/(4\pi);
$$

the densities Δu_{j0} characterize the displacements of points of the half-space boundary. The radiation conditions at infinity are satisfied identically for the displacement representations (3). We note that for the crack location and loading conditions described above, the only nonzero discontinuities are the discontinuities in the normal displacements Δu_{3k} ($k = \overline{1, K}$).

Defining the stresses near cracks by Hooke's law and taking into account boundary conditions (2), we reduce the basic problem of the dynamic theory of elasticity to the following system of K two-dimensional boundary integral equations of the type of the Helmholtz potentials for the unknown densities Δu_{3k} [9]

$$\iint\limits_{S_k} \frac{\Delta u_{3k}(\xi)}{|x_k - \xi|^5} L(x_k, \xi)\, dS_\xi + \sum_{n=1}^{K} (1 - \delta_{nk}) \iint\limits_{S_n} \frac{\Delta u_{3n}(\xi)}{|x_{nk} - \xi|^5} L(x_{nk}, \xi)\, dS_\xi$$

$$+ 2 \sum_{n=1}^{K} \iint\limits_{S_n} \Delta u_{3n}(\xi) \int\limits_0^\infty \frac{\tau R_2(\tau)}{T(\tau)} \Omega(x_{nk}, \xi, \tau)\, d\tau\, dS_\xi = \frac{\omega_2^2}{4G} N_{3k}(x_k, \omega), \qquad k = \overline{1, K}. \tag{4}$$

Here $|x_{nk} - \xi| = [(2d_{nk} - x_{1nk} - \xi_1)^2 + (x_{2nk} - \xi_2)^2]^{1/2}$ and G is the shear modulus of the material. The kernels $L(x_{nk}, \xi)$ and $\Omega(x_{nk}, \xi, \tau)$ are given by

$$L(x_{nk}, \xi) = \sum_{m=1}^{2} (-1)^{m+1} V_m(|x_{nk} - \xi|) \exp\left(i\omega_m |x_{nk} - \xi|\right),$$

$$V_m(z) = 9 - 9i\omega_m z - (5\omega_2^2 - \omega_m^2) z^2 + i\omega_m (2\omega_m^2 - \omega_2^2) z^3 + \delta_{1m} (2\omega_1^2 - \omega_2^2)^2 z^4/4,$$

$$\Omega(x_{nk}, \xi, \tau) = \sum_{m,l=1}^{2} (-1)^{m+1} \exp\left[-b_1 R_m(\tau) - b_2 R_l(\tau)\right][\delta_{1m}\delta_{1l}\Omega_0 + (1 - \delta_{ml})\Omega_1 + \delta_{2m}\delta_{2l}\Omega_2],$$

$$\Omega_k = [\mu\omega_2^2/(2(1-\mu))]^{2-k} (\tau/b_3)^k J_k(\tau b_3), \qquad k = \overline{0, 2},$$

$$T(\tau) = R_1(\tau) R_2(\tau) - \tau^2, \qquad R_j(\tau) = \sqrt{\tau^2 - \omega_j^2}, \qquad j = 1, 2,$$

$$b_1 = |d - x_{1nk}|, \qquad b_2 = |d - \xi_1|, \qquad b_3 = |x_{2nk} - \xi_2|.$$

Here $J_k(y)$ is a Bessel function of the kth order and a real argument. The infinite integral in these equations refers to the boundary conditions at the infinite boundary S_0 of the half-space. A distinctive feature of the equations obtained is that the integration is performed only over the finite domains of the cracks S_k, which is essential for numerical solution.

The first term in (4) contains singularities. One can easily verify this by letting ω_1 and ω_2 tend to zero and expanding the kernel $L(x_k, \xi)$ in a series:

$$\frac{L(x_k, \xi)}{|x_k - \xi|^5} = \frac{1}{|x_k - \xi|^3} + \frac{A\omega_2^2}{|x_k - \xi|} + F(x_k, \xi).$$

Here $F(x_k, \xi) = L(x_k, \xi)/|x_k - \xi|^5 - 1/|x_k - \xi|^3 - A\omega_2^2/|x_k - \xi|$ and $A = (1 - \mu)(3 - 4\gamma^2 + 3\gamma^4)/4$. Thus, the BIE (4) contain strong singularities of the form $|x_k - \xi|^{-3}$ and belong to the class of hypersingular equations. It is known [8] that the unique solutions of these equations exist in the class of functions that vanish at the contours of the domains S_k. To construct a regular analog of BIE (4), we write the densities as [8]

$$\Delta u_{3k}(\xi) = \sqrt{a^2 - \xi_1^2 - \xi_2^2}\, \Psi_{3k}(\xi), \qquad k = \overline{1, K}, \tag{5}$$

where $\Psi_{3k}(\xi)$ are continuously differentiable functions in the domains S_k. The representations (5) satisfy the condition the displacements are continuous in passing through the crack contours. Bearing the foregoing in mind, we regularize the first term in Eqs. (4) as [10]

$$\iint\limits_{S_k} \frac{\Delta u_{3k}(\xi)}{|x_k - \xi|^5} L(x_k, \xi)\, dS_\xi = \sum_{l=0}^{2} \frac{1}{l!} \sum_{i=0}^{2-l} \frac{1}{i!} [I_{li}^3(x_k) - I_{li}^{3\varepsilon}(x_k)] \frac{\partial^{l+i} \Psi_{3k}(x_k)}{\partial x_{1k}^l\, \partial x_{2k}^i}$$

$$+ A\omega_2^2 [I_{00}^1(x_k) - I_{00}^{1\varepsilon}(x_k)] \Psi_{3k}(x_k) + \iint\limits_{S_k^\varepsilon} \frac{\sqrt{a^2 - \xi_1^2 - \xi_2^2}\, L(x_k, \xi)}{|x_k - \xi|^5} \Psi_{3k}(\xi)\, dS_\xi.$$

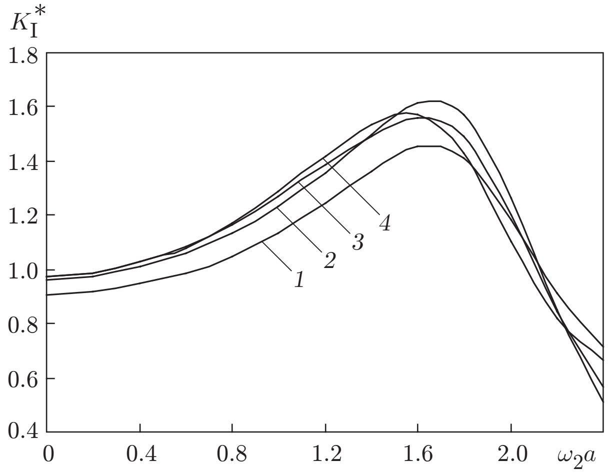

Fig. 2. Frequency dependences of K_{I}^* on the circumferential coordinates of a point inside the crack contour for $d = 1.3a$, $d_{12} = 3.0a$, and $\varphi = 0$ (1), 90° (2), 180° (3), and 270° (4).

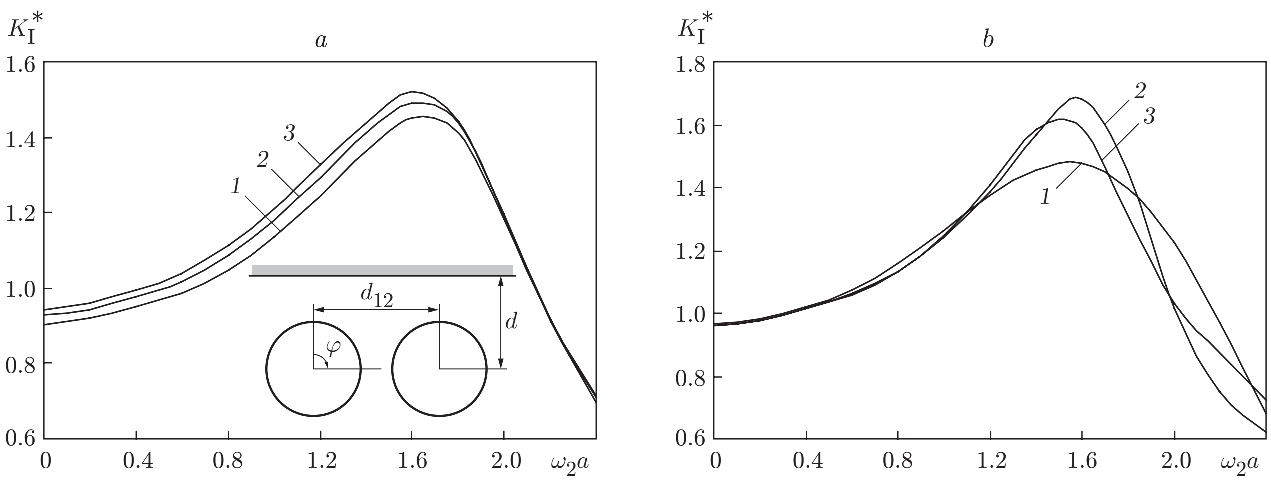

Fig. 3. Frequency dependences of K_{I}^* at fixed points of the crack contour on the depth of crack location and intercrack spacing: (a) $\varphi = 0$, $d_{12} = 3.0a$, and $d = 1.2a$ (1), $1.3a$ (2), and $1.4a$ (3); (b) $\varphi = 90^\circ$, $d = 1.4a$, and $d_{12} = 2.4a$ (1), $4.0a$ (2), and $5.0a$ (3).

The integrals $I_{li}^n(x_k)$ are given by

$$I_{li}^n(x_k) = \iint\limits_{S_k} \frac{(\xi_1 - x_{1k})^l (\xi_2 - x_{2k})^i}{|x_k - \xi|^n} \sqrt{a^2 - \xi_1^2 - \xi_2^2}\, dS_\xi$$

and evaluated in analytic form [8]. The integrals $I_{li}^{n\varepsilon}(x_k)$ differ from $I_{li}^n(x_k)$ in that the integration domain S_k^ε is obtained from S_k by eliminating of a circle of arbitrarily small radius ε centered at the point $\xi = x_k$. These integrals are evaluated numerically.

After the regularization, the BIE (4) reduce to a system of linear algebraic equations for unknown discrete values of the functions $\Psi_{3k}(\xi)$. The circular domains of the cracks S_k are discretized in the polar coordinate system $O_k r\varphi$ using rectangular boundary elements within which the discrete values of $\Psi_{3k}(\xi)$ are assumed to be constant.

Numerical Results. As an example, we consider two cracks subjected to tearing forces of constant amplitude $N_{3k}(x_k) = N_0$ $(k = 1, 2)$. The crack domains are discretized using 11 points in the radial direction r and 16 points in the circumferential direction φ. Poisson's ratio of the material is equal to 0.3. To evaluate the infinite integrals in the BIE (4), we divide the integration interval $(0, \infty)$ into the intervals $(0, \omega_1)$, (ω_1, ω_2), and (ω_2, ∞)

with the corresponding radiation conditions by choosing the branches of the radicals $R_j(\tau)$ $(j = 1, 2)$. Since the function $T(\tau)$ has a real root in the interval $(0, \omega_1)$, we regularize this integral in the interval before evaluation.

Using the solutions $\Psi_{3k}(r, \varphi, \omega)$, we obtain the mode I stress intensity factors at the crack contours

$$K_{\mathrm{I}k}(a, \varphi, t) = -\frac{2G\pi\sqrt{\pi a}}{1-\mu}\,\Psi_{3k}(a, \varphi, \omega)\exp(-i\omega t), \qquad k = 1, 2.$$

Figures 2 and 3 show the normalized amplitudes $K_{\mathrm{I}}^* = |K_{\mathrm{I}}|/K_{\mathrm{I}}^s$ ($K_{\mathrm{I}}^s = 2N_0\sqrt{a/\pi}$ is the static mode I stress intensity factor for a crack in an infinite body subjected to forces N_0) versus the cyclic frequency $\omega_2 a$. One can see that in the examined range of the parameter $\omega_2 a$, the amplitudes K_{I}^* increase monotonically from the static values for $\omega_2 a = 0$ to the maximum value and then decrease monotonically. The location of the points of the crack contours at which the amplitudes attain maximum values depends strongly on the depth of crack location and the intercrack distance. For a fixed distance between the defects d_{12}, the values of K_{I}^* at the points of the crack contours the closest to the half-space boundary decrease somewhat with decrease in the depth of crack location (see Fig. 3a). This indicates that the rigid clamping of the half-space boundary leads to strengthening of the half-space compared to the case of a free boundary [14]. For example, for $\omega_2 a = 1.5$, $d = 1.2a$, $d_{12} = 3a$, and $\varphi = 0$, we have $K_{\mathrm{I}}^* = 1.45$ and 1.56 for half-spaces with clamped and free boundaries, respectively. In the case of a single circular crack, these values are equal to 1.35 and 1.62, respectively, and for an infinite body, the amplitude is 1.51 [4]. As the distance between the cracks decreases, the dependence of the maximum amplitudes K_{I} on the distance has a wavy nature (see Fig. 3b).This effect is also observed for infinite [2, 3, 10] and semi-infinite [14] bodies with cracks subjected to harmonic loads. As the depth of defect location increases, the values of K_{I}^* tend to those in an infinite body.

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